THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

# MATH4030 Differential Geometry
## 31 October, 2024 Tutorial

1. (From exercise 3-2.5 of [Car16]) Let $S$ be an orientable surface and let $e_1, e_2$ be the principal directions of $S$ at the point $p \in S$ with principal curvatures $\kappa_1, \kappa_2$. Show the following: Hint: write everything in the basis $\{e_1, e_2\}$ of $T_pS$.

   (a) Let $v \in T_pS$ be a unit vector and let $\theta$ be the angle from $e_1$ to $v$ in the orientation of $T_pS$. Show that the normal curvature $\kappa_N$ along $v$ is given by

   $$\kappa_N = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta.$$

   Recall $\kappa_N$ at $p$ along $v \in T_pS$ is given by $\kappa_N = \langle -dN_p(v), v \rangle$.

   This is known classically as the *Euler formula*.

   (b) Show that the mean curvature $H$ at $p \in S$ is given by

   $$H = \frac{1}{\pi}\int_0^\pi \kappa_N(\theta)d\theta,$$

   where $\kappa_N(\theta)$ is the normal curvature at $p$ along a direction making an angle $\theta$ with a fixed direction.

2. Show that an orientable surface $S$ that is compact (closed and bounded) has at least one elliptic point.

3. Show that there exists no compact (closed and bounded) minimal surfaces in $\mathbb{R}^3$. A surface is called minimal if its mean curvature $H$ vanishes everywhere.

4. (Exercise 3-2.17 of [Car16]) Suppose $S$ is a regular surface with orientation $N$ so that $dN_p \neq 0$ for all $p \in S$. If the mean curvature $H$ vanishes on $S$ and $S$ contains no planar points, show that the Gauss map $dN_p : T_pS \to T_pS$ satisfies

   $$\langle dN_p(v), dN_p(w)\rangle = -K_p\langle v, w\rangle$$

   for all $p \in S$ and $v, w \in T_pS$ and where $K_p$ denotes the Gaussian curvature of $S$ at $p$.

   Hint: write everything in the basis $\{e_1, e_2\}$ of $T_pS$.

1. (From exercise 3-2.5 of [Car16]) Let $S$ be an orientable surface and let $e_1, e_2$ be the principal directions of $S$ at the point $p \in S$ with principal curvatures $\kappa_1, \kappa_2$. Show the following:

   (a) Let $v \in T_pS$ be a unit vector and let $\theta$ be the angle from $e_1$ to $v$ in the orientation of $T_pS$. Show that the normal curvature $\kappa_N$ along $v$ is given by

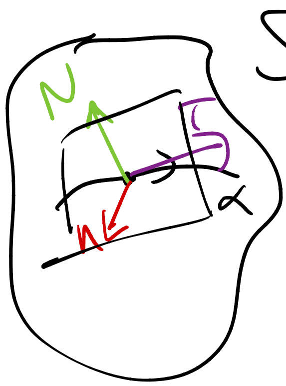
   $$\kappa_N = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta.$$

   This is known classically as the *Euler formula*.

   (b) Show that the mean curvature $H$ at $p \in S$ is given by

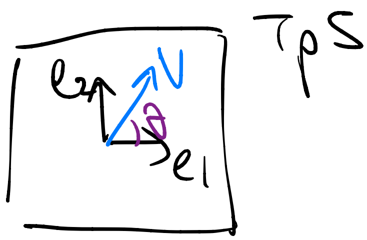
   $$H = \frac{1}{\pi}\int_0^\pi \kappa_N(\theta)d\theta,$$

   where $\kappa_N(\theta)$ is the normal curvature at $p$ along a direction making an angle $\theta$ with a fixed direction.




S let $n = N \times T$, then $\{n, T, N\}$ span $\mathbb{R}^3$.

we can write $\alpha''(0) = K_g n + K_N N$.




$V = \cos\theta e_1 + \sin\theta e_2$

$$\begin{aligned}
K_N &= \langle -dN_p(v), v \rangle \\
&= \langle -dN_p(\cos\theta e_1 + \sin\theta e_2), \cos\theta e_1 + \sin\theta e_2 \rangle. \\
&= \langle \cos\theta(-dN_p(e_1)) + \sin\theta(-dN_p(e_2)), \cos\theta e_1 + \sin\theta e_2 \rangle \\
&= \langle \cos\theta K_1 e_1 + \sin\theta K_2 e_2, \cos\theta e_1 + \sin\theta e_2 \rangle. \\
&= K_1 \cos^2\theta + K_2 \sin^2\theta
\end{aligned}$$

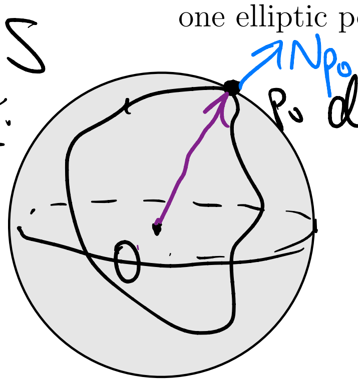
b) $\frac{1}{\pi}\int_0^{\pi} k_N(\theta)\,d\theta = \frac{1}{\pi}\int_0^{\pi} k_1\cos^2\theta + k_2\sin^2\theta\,d\theta$

$$= \frac{1}{\pi}\int_0^{\pi} ((k_1 (\frac{1+\cos(2\theta)}{2}) + k_2 (\frac{1-\cos(2\theta)}{2})) \,d\theta$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2}(k_1+k_2)\right) = \frac{1}{2}(k_1+k_2) = H.$$

2. Show that an orientable surface $S$ that is compact (closed and bounded) has at least one elliptic point.

Pf:




$p_0$ define $f : S \to \mathbb{R}$, by $f(p) = \|p\|^2$.

Since $S$ is compact, $f$ attains maximum at some point $p_0 \in S$. Let $\alpha$ be a regular curve in $S$ s.t. $\alpha(0) = p_0$.

So by 2nd derivative test,

$$\frac{d}{dt}\Big|_{t=0} f(\alpha(t)) = 0 \Rightarrow 2\langle \alpha'(0), \alpha(0)\rangle = 0.$$

$$\frac{d^2}{dt^2}\Big|_{t=0} f(\alpha(t)) \leq 0. \Rightarrow \langle \alpha''(0), \alpha(0)\rangle + \underbrace{\|\alpha'(0)\|}_{=1} \leq 0 \quad (*)$$

at $p_0$, $N_{p_0} = \frac{\alpha(0)}{\|\alpha(0)\|}$. Then $(*)$, we have

$$0 \geq \langle \alpha''(0), \alpha(0)\rangle + 1$$

$$= \langle \alpha''(0), \|\alpha(0)\| N_{p_0}\rangle + 1.$$

So we have $N_{p_0} \perp \alpha'(0)$, so

$$0 = \frac{d}{dt}\langle \alpha'(0), N_{p_0}\rangle = \langle \alpha''(0), N_{p_0}\rangle + \langle \alpha'(0), dN_{p_0}(\alpha'(0))\rangle$$

$$= \langle \alpha'(0), -dN_{p_0}(\alpha'(0))\rangle + 1$$

So what we get is

$h(\alpha'(0)) \leq \frac{1}{\|\alpha(0)\|}$. where $h$ is the second fundamental form of $S$ at $p_0$.

$k_1 \alpha_1'^2 + k_2 \alpha_2'^2 \leq -\frac{1}{\|\alpha(0)\|} < 0$ where $\alpha_1', \alpha_2'$ are the components of $\alpha'(0)$ in the basis $\{e_1, e_2\}$ of principal directions.

So $k_1, k_2$ have same sign and $K_{p_0} = k_1 k_2 > 0$.

//.

3. Show that there exists no compact (closed and bounded) minimal surfaces in $\mathbb{R}^3$. A surface is called minimal if its mean curvature $H$ vanishes everywhere.

Pf: $H = \frac{1}{2}(k_1 + k_2)$. So if $H \equiv 0$ for all $p$,
then $k_1, k_2$ have opposite signs everywhere.
But by previous question, there must be at least one elliptic point, i.e., $p_0$ where $k_1(p_0), k_2(p_0)$ have same sign. Contradiction. //

4. (Exercise 3-2.17 of [Car16]) Suppose $S$ is a regular surface with orientation $N$ so that $dN_p \neq 0$ for all $p \in S$. If the mean curvature $H$ vanishes on $S$ and $S$ contains no planar points, show that the Gauss map $dN_p : T_pS \to T_pS$ satisfies

$$\langle dN_p(v), dN_p(w)\rangle = -K_p\langle v, w\rangle$$

for all $p \in S$ and $v, w \in T_pS$ and where $K_p$ denotes the Gaussian curvature of $S$ at $p$.

Pf: Let $p \in S$ and $k_1, k_2$ be the principal curvatures at $p$, with principal directions $e_1, e_2$. Since $H = 0$ at $p$,

$$\tfrac{1}{2}(k_1 + k_2) = 0 \Rightarrow k_2 = -k_1.$$

So $K_p = k_1 \cdot k_2 = -k_1^2$.

Let $v, w \in T_pS$. Write $v = a_1e_1 + a_2e_2$, $w = b_1e_1 + b_2e_2$

$$\langle dN_p(v), dN_p(w)\rangle = \langle -dN_p(v), -dN_p(w)\rangle$$

$$= \langle -dN_p(a_1e_1 + a_2e_2), -dN_p(b_1e_1 + b_2e_2)\rangle$$

$$= \langle a_1k_1e_1 + a_2k_2e_2, b_1k_1e_1 + b_2k_2e_2\rangle$$

$$= \langle a_1k_1e_1 - a_2k_1e_2, b_1k_1e_1 - b_2k_1e_2\rangle$$

$$= k_1^2\langle a_1e_1 - a_2e_2, b_1e_1 - b_2e_2\rangle$$

$$= k_1^2\big(a_1b_1\langle e_1, e_2\rangle - a_1b_2\langle e_1, e_2\rangle + a_2b_1\langle e_2, e_1\rangle + a_2b_2\langle e_2, e_2\rangle\big)$$

$$\langle v, w\rangle = \langle a_1e_1 + a_2e_2, b_1e_1 + b_2e_2\rangle$$

$$= -K_p\langle v, w\rangle.$$